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SOME REMARKS ON THE ODD HADWIGER'S CONJECTURE

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We say that H has an odd complete minor of order at least l if there are l vertex disjoint trees in H such that every two of them are joined by an edge, and in addition, all the vertices of trees are two-colored in such a way that the edges within the trees are bichromatic, but the edges between trees are monochromatic.

Gerards and Seymour conjectured that if a graph has no odd complete minor of order l, then it is $(l-1)$-colorable. This is substantially stronger than the well-known conjecture of Hadwiger. Recently, Geelen et al. proved that there exists a constant c such that any graph with no odd K_k-minor is $ck\sqrt{\log k}$-colorable. However, it is not known if there exists an absolute constant c such that any graph with no odd K_k-minor is ck-colorable.

Motivated by these facts, in this paper, we shall first prove that, for any k, there exists a constant $f(k)$ such that every $(496k+13)$-connected graph with at least $f(k)$ vertices has either an odd complete minor of size at least k or a vertex set X of order at most $8k$ such that $G-X$ is bipartite. Since any bipartite graph does not contain an odd complete minor of size at least three, the second condition is necessary. This is an analogous result of Böhme et al.

We also prove that every graph G on n vertices has an odd complete minor of size at least $n/2\alpha(G)-1$, where $\alpha(G)$ denotes the independence number of G. This is an analogous result of Duchet and Meyniel. We obtain a better result for the case $\alpha(G)=3$.

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1. Introduction

In this paper, all graphs are simple (that is, they have no loops or parallel edges) and finite; K_n and $K_{n,m}$ denote, respectively, the complete graph on n vertices and the complete bipartite graph such that one partite set has n vertices and the other partite set has m vertices. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. An H *minor* is a minor isomorphic to H.

Our research is motivated by Hadwiger's Conjecture from 1943 which suggests a far-reaching generalization of the Four Color Theorem [1,2,20] and is considered by many as one of the deepest open problems in graph theory. Hadwiger's Conjecture states the following.

Conjecture 1.1. For all $k \geq 1$, every k-chromatic graph has a K_k minor.

Conjecture 1.1 is trivially true for $k \leq 3$, and reasonably easy for $k = 4$, as shown by Dirac [6] and Hadwiger himself [8]. However, for $k \geq 5$, Conjecture 1.1 implies the Four Color Theorem. In 1937, Wagner [24] proved that the case $k = 5$ of Conjecture 1.1 is, in fact, equivalent to the Four Color Theorem. In 1993, Robertson, Seymour and Thomas [21] proved that a minimal counterexample to the case $k = 6$ is a graph G which has a vertex v such that $G - v$ is planar. By the Four Color Theorem, this implies Conjecture 1.1 for $k = 6$. Hence the cases $k = 5, 6$ are each equivalent to the Four Color Theorem [1,2,20]. Conjecture 1.1 is open for $k \geq 7$. For the case $k = 7$, Toft and the first author [13] proved that any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor. Recently, the first author [12] proved that any 7-chromatic graph has K_7 or $K_{3,5}$ as a minor.

It is not known if there exists an absolute constant c such that any ck-chromatic graph has a K_k minor. So far, it is known that there exists a constant c such that any $ck\sqrt{\log k}$-chromatic graph has a K_k minor. This follows from the results of Kostochka [16,17] or Thomason [22,23].

Recently, the concept of an "odd minor" is paid much attention by many researchers. Let us recall the definition of odd minor. We say that H has an *odd complete minor* of size at least l if there are l vertex disjoint trees in H such that every two of them are joined by an edge, and in addition, all the vertices of trees are two-colored in such a way that the edges within the trees are bichromatic, but the edges between trees are monochromatic. We say that H has an *odd* K_l *minor*, denoted by $H > K_l^o$, if H has an odd complete minor of size at least l. It is easy to see that any graph that has an odd K_l minor certainly contains K_l as a minor.

Gerards and Seymour (see [11], page 115.) conjectured the following.

Conjecture 1.2. For all $l \geq 1$, every graph with no odd K_{l+1} minor is l-colorable.

This conjecture is substantially stronger than Hadwiger's Conjecture. Again, Conjecture 1.2 is trivially true when $l=1,2$. In fact, when $l=2$, Conjecture 1.2 states that if a graph has no odd cycles, then it is 2-colorable. One easily checks that this is true because any graph with no odd cycles must be bipartite. The case $l=3$ was proved by Catlin [4]. Recently, Guenin [10] announced a solution of the case $l=4$. This result would imply the Four Color Theorem because a graph having an odd K_5-minor certainly contains a K_5-minor. Conjecture 1.2 is open for $l \geq 5$.

Recently, Geelen et al. [9] proved that there exists a constant c such that any graph with no odd K_k-minor is $ck\sqrt{\log k}$-colorable. This is an analogue of the results of Kostochka [16, 17] or Thomason [22, 23]. It is not known if there exists an absolute constant c such that any graph with no odd K_k-minor is ck-colorable. However, if we just consider the connectivity and large graphs, then the situation changes. Recently, the following result was proved in [3].

Theorem 1.3. *For any integer t, there exists a constant $N(t)$ such that every $2t$-connected graph with minimum degree at least $31t/2$ and with at least $N(t)$ vertices contains a K_t minor.*

Actually, the main result proved in [3] is stronger:

Theorem 1.4. *For any integers t, s and n, there exists a constant $N(t,s,n)$ such that every $(3t+2)$-connected graph of minimum degree at least $\frac{31}{2}(t+1)-3$ and with at least $N(t,s,n)$ vertices contains either $K_{t,sn}$ as a topological minor or a minor isomorphic to s disjoint copies of $K_{t,n}$.*

It is necessary to include the possibility of having $K_{t,sn}$ as a subdivision since G could be a complete bipartite graph $K_{\frac{31}{2}(t+1)-3,N}$, where N could be arbitrarily large. Also G must be "large"; if G is not "large", then only the connectivity of order $\Theta(t\sqrt{\log t})$ forces the presence of K_t-minors. Motivated by Theorem 1.3, our first main result of this paper is the following.

Theorem 1.5. *For any k, there exists a constant $f(k)$ such that every $(496k+13)$-connected graph with at least $f(k)$ vertices has either an odd K_k-minor or a vertex set X of order at most $8k$ such that $G-X$ is bipartite.*

This result is an analogue of Theorem 1.3. Note that since G could be bipartite, and any bipartite graph does not contain an odd complete minor of size at least 3, the second condition is necessary. This is the first result that the linear connectivity forces either the existence of an odd complete minor or a small odd cycle cover (that is, there is a vertex set X of bounded number of vertices in G such that $G-X$ is bipartite, namely, all odd cycles in G must hit at least one vertex in X).

There is another direction to think about Hadwiger's Conjecture. This approach concerns the relation between the independence number and the chromatic number of a graph.

Let $\chi(G)$ denote the chromatic number of a graph G. In a $\chi(G)$-coloring of G each color-class has size at most $\alpha(G)$, the size of a maximum independent set of G. Hence $\chi(G) \geq |V(G)|/\alpha(G)$. Conjecture 1.1 suggests that any graph G has the complete graph $K_{\chi(G)}$ as a minor. Hence, by the above inequality, Conjecture 1.1 implies the following.

Conjecture 1.6. Any graph G on n vertices has a $K_{\lceil n/\alpha(G)\rceil}$ minor.

This conjecture seems weaker than Hadwiger's Conjecture, however, when $\alpha(G) = 2$, these two conjectures are equivalent [18]. Conjecture 1.6 was explicitly stated and put into context by Woodall [25]. In 1982, Duchet and Meyniel [7] proved the following result.

Theorem 1.7. *Any graph G on n vertices has a $K_{\lceil n/2\alpha(G)-1\rceil}$ minor.*

This result was recently improved by Plummer, Toft and the first author [14] and further improved by the authors [15] (see Section 5 for more details).

For $\alpha(G) = 2$, Theorem 1.7 states that any graph on n vertices has a complete minor of size at least $n/3$. It seems quite hard to improve the constant $1/3$. Seymour and Mader independently asked if it is possible to obtain a larger constant than $1/3$ for this case. As far as we know no such improvement has been achieved so far.

Motivated by these facts, our second result is the following.

Theorem 1.8. *Any graph G on n vertices has an odd $K_{\lceil n/2\alpha(G)-1\rceil}$ minor.*

In Section 4, we prove the following slightly stronger result for the case $\alpha(G)=3$, which is an analogue of a result in [15].

Theorem 1.9. *Any graph G on n vertices with $\alpha(G)=3$ has an odd $K_{\lceil n/4\rceil}$ minor.*

We need to introduce more notation. For a graph G we use $|G|$ to denote the number of vertices of G. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$ or simply $N(x)$, and $d_G(x) = |N_G(x)|$ is the degree of x in G. For a subset S of $V(G)$, the subgraph induced by S is denoted by $G[S]$ and $G - S = G[V(G) - S]$. For a subgraph H of G, $G-H=G[V(G)-V(H)]$. For graph-theoretic terminology not explained in this paper, we refer the reader to [5].

2. Proof of Theorem 1.5

Before we prove Theorem 1.5, we need some definitions. A complete minor of order l can be thought of l vertex disjoint trees, every two of which are joined by an edge. We call such a minor *even* if the union of these trees is bipartite. We call such a minor *odd* if its vertices can be two-colored so that the edges in the trees are bichromatic but the edges between two disjoint trees are monochromatic. Geelen et al. [9] proved the following result.

Theorem 2.1. *Suppose G is $(8k+2)$-connected. If G has an even complete minor of order at least $16k$, then either G has an odd complete minor of order k or G has a vertex set X with $|X|<8k$ such that $G-X$ is bipartite.*

Note that Theorem 2.1 and Theorem 8 in [9] are slightly different, but if we assume that G is $(8k+2)$-connected, then Theorem 8 in [9] implies Theorem 2.1. By Theorem 1.4 applied to the case $t=32k$, $s=1$ and $n=(16k-1)\binom{32k}{16k}+1$, the following holds:

> For any k, there exists a constant $f(k)$ such that any $(496k+13)$-connected graph with at least $f(k)$ vertices has a $K_{32k,(16k-1)\binom{32k}{16k}+1}$-minor.
>
> We can think of a $K_{32k,(16k-1)\binom{32k}{16k}+1}$-minor as follows:
>
> There are $32k+(16k-1)\binom{32k}{16k}+1$ disjoint trees $T_1,\dots,T_{32k},T'_1,\dots,T'_{(16k-1)\binom{32k}{16k}+1}$ such that there is an edge between T_i and T'_j for any i,j with $1\le i\le 32k$ and $1\le j\le(16k-1)\binom{32k}{16k}+1$.

We first two-color (using colors 1 and 2) the trees $T_1,\dots,T_{32k}$ such that each T_i is bichromatic. Then for each j, we two-color T'_j in such a way that T'_j is bichromatic and there are at least $16k$ bichromatic edges between T'_j and $\bigcup_{i=1}^{32k}T_i$, for $1\le j\le(16k-1)\binom{32k}{16k}+1$. This is possible since we have two choices for two-coloring of T'_j. Then by the Pigeonhole Principle, there are $16k$ disjoint trees in $\{T_1,\dots,T_{32k}\}$, say trees $T_1,\dots,T_{16k}$, and there are $16k$ disjoint trees in $\{T'_1,\dots,T'_{(16k-1)\binom{32k}{16k}+1}\}$, say trees $T'_1,\dots,T'_{16k}$, in such a way that each edge between T_i and T'_j is bichromatic for $i=1,\dots,16k$ and $j=1,\dots,16k$.

Now let $T^*_i=T_i\cup T'_i$, where $i=1,\dots,16k$. Clearly $\bigcup_{i=1}^{16k}T^*_i$ is bipartite and forms an even complete minor of order $16k$ in G. By Theorem 2.1, either G has an odd complete minor of order k or G has a vertex set X of order at most $8k$ such that $G-X$ is bipartite. This completes the proof of Theorem 1.5. ∎

3. Proof of Theorem 1.8

We shall prove that any non-empty graph G has an odd $K_{\lceil |G|/2\alpha(G)-1\rceil}$ minor.

The proof of Theorem 1.8 is by induction on $\alpha(G)$. Clearly, the statement is true when $\alpha(G)=1$. So we may assume that $\alpha(G)\geq 2$. Throughout this section we assume that G is a graph with $|G|=n$ and $\alpha=\alpha(G)\geq 2$ but G is not contractible to an odd $K_{\lceil n/2\alpha-1\rceil}$ minor. We may assume that $n\geq 4\alpha-2$, otherwise $G>K^o_{\lceil n/2\alpha-1\rceil}$, a contradiction.

Claim 1. G is connected.

Proof. Suppose G is disconnected. Let G_1 and G_2 be two subgraphs of G such that $G_1\cup G_2=G$ and $V(G_1)\cap V(G_2)=\emptyset$. Let $\alpha_i:=\alpha(G_i)$ and $n_i:=|G_i|$ for $i=1,2$. Then $n=n_1+n_2$ and $\alpha=\alpha_1+\alpha_2$. By induction $G_i>K^o_{\lceil \frac{n_i}{2\alpha_i-1}\rceil}$ for $i=1,2$. Since G is not contractible to an odd $K_{\lceil n/2\alpha-1\rceil}$ minor, we have

$$n_i<\frac{n(2\alpha_i-1)}{2\alpha-1},$$

for $i=1,2$. It follows that $n=n_1+n_2<\frac{n(2\alpha-2)}{2\alpha-1}$, which is impossible. This completes the proof of Claim 1. ∎

Claim 2. For any $x\in V(G)$, $N(x)$ is not complete.

Proof. Suppose for a contradiction that there exists $x\in V(G)$ such that $N(x)$ is complete. Then $|N[x]|<n/2\alpha-1$. Note that $\alpha(G-N[x])\leq\alpha-1$. By induction, $G-N[x]>K^o_{n-|N[x]|/2(\alpha-1)-1}>K^o_{n/2\alpha-1}$, a contradiction. This completes the proof of Claim 2. ∎

Let x be a vertex of G. Let I_0 be a maximum independent set of $N(x)$. By Claim 2, $|I_0|\geq 2$. We now build a set $S\subset V(G)$ starting from $I_0\cup\{x\}$. The obtained S satisfies:

(i) $I_0\cup\{x\}\subseteq S\subseteq V(G)$;
(ii) $|S|\leq 2\alpha-1$;
(iii) There exists $I\subset S$ such that $I_0\subset I$ and I is independent in G and every vertex in $G-S$ has a neighbor in I.

Let us briefly sketch how to build S. We first partition the vertices of G as follows. Let $L_1=\{x\}$ and $L_2=I_0$. Now for some integer $i\geq 3$, we may assume $L_1,\ldots,L_{i-1}$ have been constructed. Define $L_i=N(L_{i-1})$ if i is odd and $L_i=I(N(L_{i-1}))$ if i is even, where $N(L_{i-1})=(\bigcup_{u\in L_{i-1}}N_G(u))-(L_{i-1}\cup L_{i-2})$ and $I(N(L_{i-1}))$ denotes a maximum independent set of $G[N(L_{i-1})]$. Since G is finite, this procedure has to stop eventually. We may assume that the above procedure stops at step k. Since G is connected by Claim 1, we have $V(G) = \cup_{i=1}^{k} L_i$. Let $t = \lfloor \frac{k}{2} \rfloor$. For any $1 \leq j \leq t$, let E_{2j} be a set of edges between L_{2j} and L_{2j-1} such that each vertex $v \in L_{2j}$ is incident to exactly one edge in E_{2j}. Let $L'_{2j-1} = V(E_{2j}) \cap L_{2j-1}$. Clearly, $|L'_{2j-1}| \leq |L_{2j}|$ and $L'_1 = L_1$. Let $S = (\bigcup_{j=1}^{t} L'_{2j-1}) \cup (\bigcup_{j=1}^{t} L_{2j})$. Let $I = \bigcup_{j=1}^{t} L_{2j}$. Clearly, I is an independent set in G. Since $|L'_1| < |L_2|$, from the above construction, we see that $|S| \leq 2|I| - 1$. Since $|I| \leq \alpha$ we have $|S| \leq 2\alpha - 1$. It can be easily checked that S satisfies conditions (i)–(iii).

From the above construction, let E'_{2j-1} be a set of edges between L'_{2j-1} and L_{2j-2} such that each vertex $v \in L'_{2j-1}$ is incident to exactly one edge in E'_{2j-1}, where $2 \leq j \leq t$. Let T be the subgraph of G with vertex set S and edge set $(\bigcup_{j=2}^{t} L'_{2j-1}) \cup (\bigcup_{j=1}^{t} L_{2j})$. Clearly, T is an induced tree. By induction, $G - T$ has an odd $K_{\lceil n - |S|/2\alpha - 1 \rceil}$ minor. Now color the vertices of I by color 1, and the rest vertices of T by color 2. It follows that G has an odd $K_{\lceil n/2\alpha - 1 \rceil}$ minor, a contradiction. This completes the proof of Theorem 1.8. ∎

4. Proof of Theorem 1.9

Suppose there exists a graph G with $\alpha(G) = 3$ such that G is not contractible to an odd complete minor of size at least $|G|/4$. Among all such graphs we choose G with $|G| = n$ minimum.

Claim 1. G is connected.

Proof. Suppose G is disconnected. Since $\alpha(G) = 3$, at least one component of G is complete. Let G_1 be a component of G which is complete. Thus $|G_1| < \frac{n}{4}$. Let $G_2 := G - G_1$. Then $|G_2| \geq \frac{3n}{4}$. Since $\alpha(G) = 3$, we see that $\alpha(G_2) \leq 2$. By Theorem 1.8, H_2 contains an odd complete minor of size $\frac{|H_2|}{3} > \frac{n}{4}$, a contradiction. ∎

Since $\alpha(G) = 3$, by Claim 1, G contains at least one induced 2-edge path. Let $P_1, P_2, \ldots, P_t$ be pairwise disjoint induced 2-edge paths in G with t maximum. Let $V(P_k) = \{x_k, y_k, z_k\}$, where $1 \leq k \leq t$, and $x_k y_k, y_k z_k \in E(G)$ but $x_k z_k \notin E(G)$.

Claim 2. $t < \frac{n}{4}$.

Proof. For each P_k, where $1 \leq k \leq t$, let us color x_k, z_k by color 1 and y_k by color 2. Since $\alpha(G) = 3$, it is easy to see that for any i, j with $i \neq j$, there is a monochromatic edge between P_i and P_j. We see that G has an odd complete minor of size t. Thus $t < \frac{n}{4}$, as claimed. ∎

Since $\alpha(G)=3$, $G-\bigcup_{k=1}^{t} P_k$ has at most three components. By the choice of t, each component of $G-\bigcup_{k=1}^{t} P_k$ is complete. Let $G-\bigcup_{k=1}^{t} P_k$ have components $C_1,\ldots,C_m$. Then $m\le 3$. Let $C_j:=\emptyset$ for $m<j\le 3$. Since $\alpha(G)=3$, we have:

Claim 3. If there exist a vertex $a\in V(C_1)\cup V(C_2)\cup V(C_3)$ and P_k such that $ax_k, az_k\notin E(G)$, then for any $b\in V(C_1)\cup V(C_2)\cup V(C_3)$ such that $ab\notin E(G)$, $bx_k\in E(G)$ or $bz_k\in E(G)$.

For $1\le i\le 3$ let T_i be the set of indices $1\le j\le t$ for which some vertex of C_i is adjacent to neither x_j nor z_j (so $T_i=\emptyset$ if $C_i=\emptyset$). By Claim 3, T_1,T_2,T_3 are disjoint. Moreover, C_i together with P_j's where $j\notin T_i$ forms an odd complete minor of size $|C_i|+t-|T_i|$ (by coloring x_j,z_j and the vertices of C_i by color 1 and y_j by color 2, where $j\notin T_i$). Thus $|C_i|+t-|T_i|<n/4$ for $i=1,2,3$. Since T_1,T_2,T_3 are disjoint and $|T_1|+|T_2|+|T_3|\le t$, by adding up these three inequalities, we obtain

$$3n/4>\sum_{i=1}^{3}(|C_i|+t-|T_i|)\ge(n-3t)+3t-t=n-t.$$

It follows that $t>n/4$, contrary to Claim 2. This completes the proof of Theorem 1.9. ∎

5. Concluding Remarks

Recently, Plummer, Toft and the first author [14] improved Theorem 1.7 as follows.

Theorem 5.1. *Any graph G on n vertices with $\alpha(G)\ge 3$ has a $K_{\lceil 2n/4\alpha(G)-3\rceil}$ minor.*

Theorem 5.1 was further improved by the authors [15].

Theorem 5.2. *Any graph G on n vertices with $\alpha(G)\ge 3$ has a $K_{\lceil n/2\alpha(G)-2\rceil}$ minor.*

It is tempting to conjecture that one can also improve Theorem 1.8 as in Theorem 5.2. For $\alpha(G)=3$, this is already done in Theorem 1.9. Perhaps Theorem 1.9 is the first step to improve Theorem 1.8, as the authors did in [15].

Reed and Seymour [19] extended Theorem 1.7 to the fractional version of Hadwiger's conjecture. It is perhaps possible to prove the following.

Conjecture 5.3. If G does not contain an odd K_{k+1}-minor, then the fractional chromatic number of G is at most $2k$.

This would generalize Theorem 1.8.

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